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Are Rain Rate Processes Self-Similar?

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We clarify the implication, with respect to self-similarity, of the facts that the probability of rain tends to increase with the size of the area of observation and that the distribution of rain rate contains an atom at zero. These properties observed in precipitation processes lead to subtle difficulties which are not compatible with self-similarity.

INTRODUCTION

A real-valued stochastic process $\{X(t),\ t \geq 0\}$ is said to be self-similar with parameter $H > 0$ ($H - ss$) if

$$\{X(\lambda t)\} \stackrel{d}{=} \{\lambda^H X(t)\} \qquad \forall\ \lambda > 0 \qquad t \geq 0 \tag{1}$$

where $\stackrel{d}{=}$ denotes equality of the finite-dimensional distributions of the two processes. In general, the parameter H need not be positive, but in order to avoid trivialities, H is restricted to be positive throughout the paper. Self-similar processes are sometimes also called scaling processes [see *O'Brien and Vervaat*, 1983; *Taqqu*, 1987].

Scaling or self-similarity is an important issue which has attracted considerable attention and has particularly inspired a certain trend in rainfall modeling. From the applications point of view the scaling property, if indeed it is found to hold, can lead to far-reaching consequences in data collection and interpretation. For example, in remote sensing via satellites the scaling property means that the parameters of interest are independent of the pixel size in the sense that knowledge of these parameters over a very small pixel is equivalent to knowledge over extremely large pixels. Two recent comprehensive review papers of scaling issues are *Waymire* [1985] and *Lovejoy and Schertzer* [1985]. *Waymire* [1985] shows that under some conditions, self-similarity can be induced by certain limiting operations applied to a process. The limit process is endowed with the self-similar property and a universal fluctuation law which can help in the estimation of rainfall probabilities. *Lovejoy and Schertzer* [1985] discuss empirical evidence of self-similarity observed in the atmosphere and leading to fat-tailed probability distributions of rainfall measurements. The present brief paper, however, raises some serious questions as to the validity and applicability of self-similarity when it comes to rain rate processes.

Intuitively, one can expect some form of "scaling" consideration to enter in modeling space-time phenomena. But while self-similarity may very well be observed to some extent, raw rain rate cannot be self-similar without, perhaps, some limiting operations as discussed by *Waymire* [1985]. The difficulty arises from two critical observations: (1) The probability of rain as a function of the area or pixel size is not in general constant. (2) The distribution of rain rate is of a mixed type, having an atom at zero. From a strictly mathematical point of view these two critical observations on the one hand and self-similarity on the other hand are, under some general conditions, contradictory. This is discussed in what follows.

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AREA AVERAGE RAIN RATE PROCESSES

Suppose we observe rainfall over a square of area A whose center is fixed at some point in space. We may increase or decrease A about the fixed point. Clearly, $A \geq 0$. Let $R(A)$ be the corresponding area average rain rate. Then A is the parameter of the process

$$\{R(A),\ A \geq 0\}$$

to which we refer as an area average rain rate process. Note that A plays the role of t above. Evidently, the probability of observing rain over our square when its area is A is

$$p_A \equiv P(R(A) > 0) \tag{2}$$

It is interesting to record p_A as a function of A, $A > 0$. Figure 1 shows the graph of $\hat{p}_A$, an estimate of p_A, as obtained from the center of the GATE data (GATE: Global atmospheric research program, Atlantic Tropical Experiment). It is seen that $\hat{p}_A$ is increasing with $A^{1/2}$, as is well expected on intuitive grounds, for a larger area has a greater chance to realize rainfall. Thus the assumption that p_A is an increasing function of A is very reasonable. But when this assumption holds, $\{R(A)\}$ cannot be self-similar.

Theorem 1

Let $\{R(A),\ A \geq 0\}$ be an area average rain rate process. If p_A is an increasing function of A, then $\{R(A)\}$ cannot be self-similar.

Proof

Suppose $\{R(A)\}$ is self-similar. Then from (1) with $A = 1$ we have

$$R(\lambda) \stackrel{d}{=} \lambda^H R(1) \qquad \forall\ \lambda > 0$$

Also, with $A = 0$, (1) implies that

$$R(0) \stackrel{d}{=} \lambda^H R(0) \qquad \forall\ \lambda > 0$$

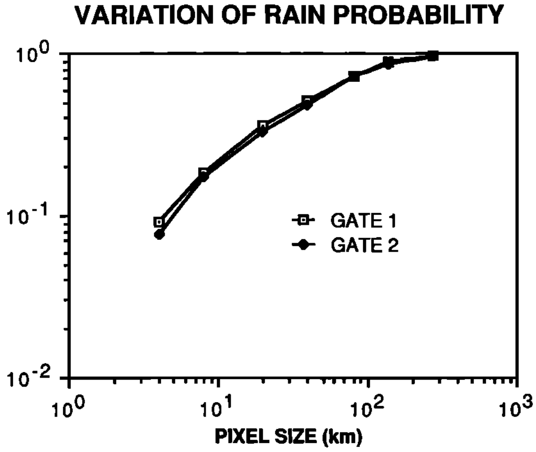

Fig. 1. Graph of $\hat{p}_A$ versus $A^{1/2}$ on a logarithmic scale.

and hence $R(0) = 0$ with probability 1. By putting these two facts together we have the representation

$$R(A) \stackrel{d}{=} A^H R(1) \qquad A \geq 0 \tag{3}$$

Therefore, for any $A > 0$,

$$P(R(A) > 0) = P(A^H R(1) > 0) = P(R(1) > 0)$$

and so p_A, being a constant for $A > 0$, cannot increase with A, which gives the desired contradiction.

General Rain Rate Processes

In the previous section we dealt with rain rate averaged over areas. In more general setups we need not restrict attention to area averages, and the parameter of R, say t, may stand for time, area, or location in space along a straight line. We will refer to $\{R(t), t \geq 0\}$ as a rain rate process with parameter t. It is convenient to refer to t as "time." In this section we assume that $\{R(t)\}$ has stationary increments. That is, $\{R(t)\}$ has stationary increments if

$$\{R(b+t) - R(b)\} \stackrel{d}{=} \{R(t) - R(0)\} \qquad \forall\, b > 0 \tag{4}$$

The assumption of stationary increments is reasonable, for example, when we observe rain rate as a function of time at a fixed location throughout a certain season, since then $R(t)$ itself may be taken as stationary. Some empirical evidence to this effect is provided by the rapidly decaying autocorrelation functions of increment processes obtained from two GATE time series as shown in Table 1.

Because it does not rain all the time, the distribution of $R(t)$ for each t contains an atom at zero. That is,

$$P(R(t) = 0) > 0 \qquad \forall\, t \tag{5}$$

Also note that rain is a recurrent event. It follows that if $\{R(t)\}$ has stationary increments, it cannot be self-similar.

Theorem 2

Let $\{R(t), t \geq 0\}$ be a rain rate process with stationary increments. Then it cannot be self-similar.

Proof

We employ a beautiful argument due to *O'Brien and Vervaat* [1983]. Suppose $\{R(t)\}$ is self-similar. Then we will show that for every $s, t > 0$,

$$P(R(t) = 0 \mid R(s) = 0) = 1$$

To achieve this goal, it suffices to show that

$$P(R(s) = 0, R(t) \neq 0) = 0$$

Then the result follows on account of the total probability formula

$$P(R(s)=0) = P(R(s)=0, R(t)\neq 0) + P(R(s)=0, R(t)=0)$$

and the definition of conditional probability.

Self-similarity implies (see (3)) that

$$q \equiv P(R(t) = 0)$$

is independent of $t > 0$. Clearly, $q > 0$. By (4) and since $R(0) = 0$ with probability 1,

$$P(R(s) = R(t)) = P(|R(s) - R(t)| = 0)$$
$$= P(R(|t-s|) = 0) = q \tag{6}$$

for all $s, t \geq 0$, and $s \neq t$. Fix t and choose an arbitrary $M > 0$. Then by (3) and the fact that rain rate is never negative,

$$P(R(t) = R(u) \neq 0)$$
$$\leq P(0 < R(t) = R(u) < M) + P(R(t) = R(u) \geq M)$$
$$\leq P(0 < R(u) < M) + P(R(t) \geq M)$$
$$\leq P(0 < R(1) < Mu^{-H}) + P(R(t) \geq M)$$

Because $H > 0$, $P(0 < R(1) < Mu^{-H}) \to 0$ as $u \to \infty$. Also, $P(R(t) \geq M)$ can be made arbitrarily small by choosing M sufficiently large. It follows that

$$\lim_{u\to\infty} P(R(t) = R(u) \neq 0) = 0 \tag{7}$$

TABLE 1. Estimated Autocorrelation of the Increment Process $z_\Delta(t) = R(t) - R(t-\Delta)$ for $\Delta = 15, 30$ min

	GATE 1		GATE 2	
k/Δ	15 min	30 min	15 min	30 min
1	−0.110	0.298	−0.013	0.537
2	−0.250	−0.231	0.088	0.104
3	0.199	0.129	0.039	0.098
4	0.082	0.224	0.030	0.067
5	0.034	0.084	0.033	0.056
6	−0.001	−0.001	0.017	0.070
7	−0.033	0.026	0.070	0.173
8	0.114	0.091	0.181	0.205
9	−0.033	0.007	−0.028	0.075
10	−0.034	0.026	0.024	−0.042
11	0.148	0.128	−0.099	−0.113
12	−0.033	0.029	−0.041	−0.103
13	−0.030	−0.034	−0.015	−0.045
14	0.032	0.037	−0.009	−0.009
15	0.031	0.099	0.019	0.010
16	0.082	0.026	−0.017	0.011
17	−0.149	−0.129	0.031	0.017
18	−0.013	−0.049	−0.017	0.000
19	0.089	0.041	−0.005	−0.027
20	−0.092	−0.038	−0.026	−0.016
21	0.027	0.015	0.030	0.006
22	0.064	0.028	−0.022	0.008
23	−0.105	−0.108	0.028	−0.006
24	−0.047	−0.091	−0.035	−0.027
25	0.037	−0.005	−0.006	−0.057

Data are obtained from two GATE time series of rain rate averaged over a fixed 280 × 280 km² pixel.

Again by total probability,

$$P(R(s) = 0, R(t) \neq 0)$$
$$= P(R(s) = 0, R(t) \neq 0, R(u) \neq 0)$$
$$+ P(R(s) = 0, R(t) \neq 0, R(u) = 0)$$
$$\leq P(R(s) = 0, R(u) \neq 0) + P(R(t) \neq 0, R(u) = 0) \quad (8)$$

The last two terms can be expressed in a more convenient form. Note that

$$P(R(s) = R(u)) = P(R(s) = R(u) = 0) + P(R(s) = R(u) \neq 0)$$
$$P(R(s) = 0) = P(R(s) = 0, R(u) \neq 0) + P(R(s) = R(u) = 0)$$

Therefore, by equating the two resulting representations for $P(R(s) = R(u) = 0)$ and by invoking (6) we have

$$P(R(s) = 0, R(u) \neq 0)$$
$$= P(R(s) = 0) + P(R(s) = R(u) \neq 0) - P(R(s) = R(u))$$
$$= q + P(R(s) = R(u) \neq 0) - q$$
$$= P(R(s) = R(u) \neq 0) \quad (9)$$

Going back to (8), equipped with (9) and taking note of (7), we obtain

$$P(R(s) = 0, R(t) \neq 0) \leq P(R(s) = R(u) \neq 0)$$
$$+ P(R(t) = R(u) \neq 0) \to 0 \qquad u \to \infty$$

But since $q > 0$, this implies that for $s, t > 0$,

$$P(R(t) = 0 \mid R(s) = 0) = 1$$

This means that if it stops raining at time s it will never rain again at any other time t which yields the desired contradiction since rain is a recurrent event.

SUMMARY

We have shown that under some very reasonable conditions, rain rate processes without any external operations such as limiting operations cannot be self-similar. Even when replacing a rain rate process by its increments, the atom at zero is inherent and is not congenial with self-similarity by an argument similar to the one above. On the other hand, nature is considerably more complicated, so that seemingly conflicting phenomena can still coexist to some degree. Empirical evidence in support of scaling in rainfall statistics perhaps falls in this category.

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